Algebra-IV, IInd Year BMath
Back Paper/Supplementary Exam
2022

Time: 3 hours

Marks: 50

(1) Let $f(x)$ be a polynomial in $F[x]$ of degree n, and let E be a splitting field of f. Show that $[E : F]$ divides $n!$. (8)

(2) (a) Prove that any finite field is isomorphic to $\mathbb{F}_{p^n}$ for some prime p and some integer $n \geq 1$.
(b) Show that the field $\mathbb{F}_{p^n}$ is the splitting field of the polynomial $x^{p^n} - x$ over $\mathbb{F}_p$.
(c) Show that the Galois group of $\mathbb{F}_{p^n}$ over $\mathbb{F}_p$ is cyclic of order n. Find a generator.
(d) List down all the subfields E of $\mathbb{F}_{p^n}$ containing $\mathbb{F}_p$. (3+3+4+5)

(3) Identify the Galois group of the splitting field E of $x^4 - 3$ over $\mathbb{Q}$. Determine the number of quadratic extensions of $\mathbb{Q}$ contained in E. (8)

(4) Let F be a field of characteristic not dividing n which contains the nth roots of unity. Then
(a) show that for any α such that $\alpha^n \in F$, the extension $F(\alpha)$ is a cyclic extension over F, of degree dividing n, and
(b) show that any cyclic extension of degree n over F is of the form $F(\alpha)$, for some α such that $\alpha^n \in F$. (4+5)

(5) Show that the Galois group of the quintic $f(x) = x^5 - 4x - 1$ over $\mathbb{Q}$ is S_5. (10)

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